Notes for Whims on Tournaments

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Abstract

This is a note for my talk *Whims on Tournaments*. This is also a presentation of one of my miserably failed but still interesting exploration starting from my sophomore year, when I first leant what a group is. The failure is that all results I worked out have already been worked out by someone else. I call my subject *Directed Set*, where the official name is *Tournament*. Here I invite you to walk along my failure from definition of a Directed Set to my two partition theorems and to some other explorations. If time permits, we might even do the King Chicken Theorem, both the original one and the one on compact Hausdorff space. I hope you have as much fun as I had.
Prerequisite: Knowing the definition of group and homomorphism is enough to enjoy most of this talk. Some basic knowledge in topology will let you enjoy the last part.

1 Introduction

The whim starts with a naive imitation about the definition of groups.

Definition 1.1. *A* ***directed set*** $(D, \wedge)$ *is a set* D *with a commutative binary operation* $\wedge$ *such that* $\forall a, b \in D$*, either* $a \wedge b = a$ *or* $a \wedge b = b$*.*

Then recall a whole bunch of definitions about relations.

Definition 1.2. *A* ***relation*** $\sim$ *on a set* S *is a subset of* $S \times S$*, the set of all ordered pairs of elements of* S*. We write* $a \sim b$ *iff* $(a, b) \in \sim$*. We also have the following terminology:*

1. *It is* ***symmetric*** *if* $\forall a, b \in S$*,* $a \sim b$ *iff* $b \sim a$*.*
2. *It is* ***anti-symmetric*** *if* $\forall a, b \in S$*,* $a \sim b$ *and* $b \sim a$ *implies* $a = b$*.*
3. *It is* ***total*** *if* $\forall a, b \in S$*, either* $a \sim b$ *or* $b \sim a$*.*
4. *It is* ***transitive*** *if* $\forall a, b, c \in S$*,* $a \sim b$ *and* $b \sim c$ *imply* $a \sim c$*.*
5. *It is* ***reflexive*** *if* $\forall a \in S$*,* $a \sim a$*.*

Definition 1.3. *We have the following types of relations:*

1. *An* ***equivalence relation*** *is a reflexive, symmetric and transitive relation.*
2. *A* ***partial order*** *is an antisymmetric and transitive relation.*
3. *A* ***total order*** *is a total, antisymmetric and transitive relation.*

So what if we scratch the transitivity from the definition, and define a "vague order" like this:

Definition 1.4. *A relation* $\sim$ *on a set* S *is called a* ***vague order*** *if it is total and antisymmetric.*

I claim that the notion of "directed set" and "set with a vague order" are actually equivalent.

Proposition 1.5. *If a set V is a set with a vague order, then it is naturally a directed set. If D is a directed set, then it has a natural vague order.*

Proof. Let V be a set with vague order $\sim$. Then we define a binary operation $\wedge$ such that $\forall a, b \in V$, $a \wedge b = b$ iff $a \sim b$, and $a \wedge b = a$ iff $b \sim a$.

Then $\forall a, b$, if $a = b$, then clearly $a \sim a$ as $\sim$ is total. Then $a \wedge a = a$. If $a \neq b$, and $a \sim b$, then we know by antisymmetry $b \nsim a$. Then $a \wedge b = b$, and $b \wedge a = b$. The case when $a \neq b$, $b \sim a$ and $a \nsim b$ is exactly the same.

Now let D be a directed set. Then we define a relation $\sim$ such that $a \sim b$ if and only if $a \wedge b = b$. Then $\forall a, b \in D$, if $a = b$, then $a \wedge a = a$, so $a \sim a$. If $a \neq b$ and $a \wedge b = b \wedge a = b$, then $a \sim b$ and $b \nsim a$. If $a \neq b$ and $a \wedge b = b \wedge a = a$, then $b \sim a$ and $a \nsim b$. So $\sim$ is a vague order. $\square$

Note that the notion of a "directed set" is also the same as the notion of a tournament, which is a set of point such that between any two points, there is an arrow pointing towards one of the two points. So in the future we could write $a \to b$ for $a \sim b$ or $a \wedge b = b$.

So now we define a directed set $(D, \wedge)$ in a similar way we define a group. Clearly $(D, \wedge)$ is hard to be a group, given that $a \wedge b = a$ or b $\forall a, b \in D$. However, let us at least figure out when is $(D, \wedge)$ a seimgroup or a monoid.

Definition 1.6. *Here are some algebraic structure related to groups:*

1. *A **semigroup** $(G, \cdot)$ is a set G with a binary associative operation.*
2. *A **monoid** is a semigroup with identity.*
3. *A **group** is a monoid where every element has an inverse.*

Proposition 1.7. *In a directed set $(D, \wedge)$, the operation $\wedge$ is associative iff the corresponding vague order is transitive. As a result, $(D, \wedge)$ is a semigroup iff the corresponding vague order is a total order.*

Proof. $\forall a, b, c \in D$, if $\wedge$ is associative, then $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. Then if $a \to b$ and $b \to c$, then $(a \wedge b) \wedge c = b \wedge c = c$, and $a \wedge (b \wedge c) = a \wedge c$, so $a \to c$, the vague order is transitive. On the other hand, suppose the vague order is transitive, $\forall a, b, c \in D$. If $b \to c$ and $b \to a$, then $(a \wedge b) \wedge c = a \wedge c = a \wedge (b \wedge c)$. If $c \to b$ and $a \to b$, then $(a \wedge b) \wedge c = b \wedge c = b = a \wedge b = a \wedge (b \wedge c)$. If $a \to b \to c$, then $(a \wedge b) \wedge c = b \wedge c = (a \wedge b) \wedge c$. If $a \leftarrow b \leftarrow c$, then $(a \wedge b) \wedge c = a \wedge c = a \wedge (b \wedge c)$. $\square$

Remark 1.8. *A directed set is a monoid iff the corresponding vague order is a total order and has a initial element.*

2 Two Partition Theorems

The case of a total order or even a partial order is not what I am interested in. There are super interesting "order theory" and "lattice theory" about them. But I am more interesting when the transitivity breaks down. This means if we have $a \to b$ and $b \to c$, then we also have $c \to a$. These three elements form a sort of cycle.

Definition 2.1. *In a directed set D, a **cycle** $(a_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ is a collection of elements in D such that $\forall i \in \mathbb{Z}/n\mathbb{Z}, a_i \to a_{i+1}$. A **path** $[a_1, ..., a_n]$ is a collection of elements in D such that $\forall i \in \{1, ..., n-1\}, a_i \to a_{i+1}$. A cycle or a path is said to be **simple** if all a_i's are distinct. Two elements a, b of D is said to be **cycle-connected** if they are contained in the same cycle. A **cycle-connected component** C of D is a maximal cycle-connected subset of D, i.e. if x is cycle-connected to some $y \in C$, then $x \in C$ as well.*

Theorem 2.2 (The First Partition Theorem or The Hamiltonian Decomposition)**.** *Any directed set has a partition into cycle-connected components. There is an induced total order on these components. And in every finite cycle-connected component there is a simple cycle going through all elements of the component.*

Proof. It's enough to show that "cycle-connected" is an equivalence relation on any directed set D. This is trivially reflexive and symmetric, so we only need to prove transitivity. If a, b are in a cycle $(b, a_1, ..., a_n)$ where $a = a_i$ for some i, and b, c are in a cycle $(b, c_1, ..., c_n)$ where $c = c_i$ for some i. Then $(a_1, ..., a_n, b, c_1, ..., c_n, b)$ is a cycle containing a, b and c, so a, c are cycle-connected. This proves the first statement.

Now let $\mathcal{C}$ be the set of these components, and $\forall A, B \in \mathcal{C}$, we say $A \to B$ iff $\exists a \in A, b \in B$, $a \to b$. I claim that this "$\to$" is a total order. $\forall A, B \in \mathcal{C}$, pick $a \in A, b \in B$, and WLOG assume $a \to b$. And suppose $\exists a' \in A, b' \in B$ such that $b' \to a'$. Then as a, a' are in a cycle $(a, c_1, ..., c_s, a', d_1, ..., d_t)$, b, b' are in a cycle $(b, e_1, ..., e_k, b', f_1, ..., f_\ell)$, therefore we have a cycle $(a, b, e_1, ..., e_k, b', a', d_1, ..., d_t)$. So a, b are cycle-connected, so $A = B$. So clearly "$\to$" is antisymmetric. It is also trivially total. Transitivity can be proved similar to antisymmetry.

We are going to prove the last statement by induction. Let $(a_1, ..., a_n)$ be a simple cycle, and let b be some element in the component not in the cycle. Then if $a_i \to b \to a_j$ for some i, j, then $\exists t$ such that $a_t \to b \to a_{t+1}$, then $(a_1, ..., a_t, b, a_{t+1}, ..., a_n)$ is a simple cycle. The same for $a_i \leftarrow b \leftarrow a_j$. So we can WLOG assume $a_i \to b$ for all i. Then pick any a_i, it is cycle-connected to b in a cycle $(b, c_1, ..., c_m)$ where a_i is some c_j. From c_1 to c_n, WLOG let the first appeared element of $\{a_1, ..., a_n\}$ be $a_1 = c_t$ for some t. Then $(b, c_1, ..., c_t = a_1, ..., a_n)$ is the desired simple cycle. To sum up, we can keep expanding the simple cycle until it includes all elements. □

Corollary 2.3. *Every tournament has a Hamiltonian path. A tournament is Hamiltonian iff it is strongly connected. (Here tournament is just a directed set, and Hamiltoniam path or cycle is a simple path or cycle going through all elements. A tournament with a Hamiltonian cycle is called hamiltonian. A tournament with only one cycle component is called strongly connected.)*

Remark 2.4. *There are several related problems:*

1. *How many Hamiltonian path are there in a given tournament? (Always odd number)*
2. *Given a strongly connected tournament, how many Hamiltonian cycle are there? (Open)*
3. *Rosenfeld's Conjecture: For $n \geq 8$, every orientation of an n-vertex path graph can be embedded into every n-vertex tournament. (Havet & Thomassé, 2000)*
4. *Sumner's Conjecture: Every orientation of an n-vertex tree is a subgraph of every $(2n-2)$-vertex tournament. (Open)*

Now recall the definition of homomorphism for groups. Immediately we have:

Definition 2.5. *Given two directed set D, E, a **homomorphism** is a map $f : D \to E$ such that $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in D$. This is called an **embedding** if f is injective. This is called a **collapsing** if f is surjective.*

Then naturally we seek to define a kernel for f. However, a directed set does not necesarrily have any identity. So what we only have are fibers, i.e. preimages of single points.

Definition 2.6. *In a directed set D, $I \subset D$ is called an **ideal** iff $\forall x \in D \setminus I$, either $x \to y$ $\forall y \in I$, or $x \leftarrow y$ $\forall y \in I$. Or simply put, elements of I behave the same way with respect to every other point in D.*

Proposition 2.7. *In a directed set D, $I \subset D$ is an ideal iff it is a fiber of some homomorphism.*

Proof. Let I be a fiber of the homomorphism $f : D \to E$, say $I = f^{-1}(a)$. Then $\forall x \in D \setminus I$, WLOG let $f(x) \to a$. Then $\forall y \in I$, $f(x \wedge y) = f(x) \wedge f(y) = a$, so $x \wedge y \in I$. As $x \notin I$, we conclude that $x \wedge y = y$. So I is indeed an ideal.

On the other hand, let I be an ideal. Let $E = (D \setminus I) \cup \{a\}$. Let us define the following directed set structure on E: $\forall x, y \in E$, if $x, y \in E \setminus \{a\} = D \setminus I$, then $x \wedge y$ is the same as in D. If one of x, y is a, say $y = a$, then $x \wedge a = x$ if $x \wedge v = x$ $\forall v \in I$, and $x \wedge a = a$ if $x \wedge v = v$ $\forall v \in I$. Then define the map $f : D \to E$ such that $f(x) = x$ if $x \in D \setminus I$, $f(x) = a$ if $x \in I$. It is easy to see that this is a homomorphism, and $I = f^{-1}(a)$ is a fiber. □

So the above Proposition points out that when we have an ideal, i.e. a cluster of points behaving the same way, then we can collapse them to one point. The result will be a surjective homomorphism from the original directed set to the one after the collapsing. And indeed all surjective homomorphisms arise this way, so I also call such a map a collapsing directly.

Now we shall see some basic properties of ideals.

Lemma 2.8. *Let A be any index set, and $\{I_a\}_{a\in A}$ be a collection of ideals. Suppose $\cap_{a\in A} I_a \neq \emptyset$, then both $\cap_{a\in A} I_a$ and $\cup_{a\in A} I_a$ are ideals. The whole set and $\emptyset$ are vacuously ideals.*

Proof. Exercise. □

Definition 2.9. *A proper ideal is called maximal if there are no proper ideals containing it other than itself.*

Proposition 2.10. *Every ideal is contained in a maximal ideal.*

Proof. With the lemme above, this is just a routine application of the Zorn's Lemma. □

Proposition 2.11. *If two distinct maximal ideals in a directed set D have non-empty inersection, then D has more than one cycle components, i.e. D is not cycle-connected.*

Proof. Let the two ideals be A, B. As $A \cap B \neq \emptyset$, $A \cup B$ is an ideal strictly containing than A. But then as A is maximal, $A \cup B = D$. Now pick some $a \in A \setminus B, b \in B \setminus A$, WLOG let $a \to b$. Then as B is an ideal, $a \to y\ \forall y \in B$, and in particular $\forall y \in B \setminus A$. Then as A is an ideal, $x \to y\ \forall x \in A, y \in B \setminus A$. So $\forall y \in B \setminus A$, if $y \to z$, then $z \notin A$, i.e. $z \in B \setminus A$. So there will never be a path from b to a, not to mention a cycle containing a, b. □

Corollary 2.12 (Second Partition Theorem)**.** *In a cycle-connected directed set, maxial ideals are disjoint.*

Definition 2.13. *A directed set where all maximal ideal are singletons is called **simple**. Then the empty directed set and the trivial directed set are not simple. The easiest simple directed set is the two-element directed set.*

Corollary 2.14 (Structure Theorem)**.** *For any directed set D, there exists a unique simple directed set P such that a surjective homomorphism from D to P exists.*

Proof. For cycle connected ones, collapsing each maximal ideal to a point, we obtain the simple directed set P. For other ones, the unique simple directed set is the two-element directed set. □

This structure theorem gives us a description about how each directed set is built. We start from a simple directed set. Then we replace one point of it with another simple directed set. Then we replace a point of the result with yet another simple directed set. Keep doing this, we will eventually build the original directed set. So if we can figure out all simple directed sets, then we can figure out all directed sets.

Proposition 2.15. *$\forall$ positive integer $n \neq 1$ or 4, there exists a simple directed set with n elements. There exists a infinite simple directed set.*

Proof. For odd n, let $D = \{a_1, ..., a_n\}$ such that $a_i \to a_j$ iff $|j - i| < \frac{n}{2}$, where $|j - i|$ is how many times i needs to increase by one in order to be j in the $\mathbb{Z}/n\mathbb{Z}$ sense. Then this is the desired simple directed set.

For even $n \geq 6$, let $n = 2k$, and let $D = \{a_1, ..., a_k, b_1, ..., b_k\}$. Let $(a_1, ..., a_k)$, $(b_k, ..., b_1)$ and $(a_1, b_1, ..., a_k, b_k)$ be cycles. Then this is the desired simple directed set.

On a circle of radius 1 on the plain, let $D = \{(cos\theta, sin\theta) : \theta \in \mathbb{Q}\}$. Note that this means θ is never a rational multiple of π excepte $\theta = 0$. Let all arrows be oriented counterclockwise, then this is the desired simple directed set. Note that this is a □

The above proposition illustrates that the simple directed sets are still pretty complicated. This is how far I went for the combinatorial aspect of directed sets

3 Directed Space

Recall that in a directed set $(D, \wedge)$, the wedge can be seen as a map $\wedge : D \times D \to D$.

Definition 3.1. *Let D be a topological space. Then $(D, \wedge)$ is a directed space if the map $\wedge$ is continuous.*

This seems to be a good definition. However, all such space is doomed to be super ugly, as addressed by the following theorem.

Theorem 3.2. *Let $(D, \wedge)$ be a directed space which is path-connected and Hausdorff. Then the vague order on it is a total order.*

Proof. We shall proceed by the following claims.

Claim 1: In any Hausdorff directed space $(D, \wedge)$, if $x \to y$, then $\exists$ disjoint open sets $U \ni x$, and $V \ni y$ such that $\wedge(U \times V) \subset V$.

Proof of Claim 1: As the space is Hausdorff, let W be a open nbhd of y disjoint from an open nbhd M of x. Then as the map $\wedge$ is continuous, $\exists$ open sets $U' \ni x, V' \ni y$ such that $\wedge(U' \times V') \subset W$. Then $x \wedge a \in W$ $\forall a \in V'$. But as $x \notin W$, we see that $x \wedge a = a$ $\forall a \in V'$ and $V' \subset W$. Let $U = U' \cap M$, $V = V'$, then the rest follows trivially.

Claim 2: All path-connected subspaces of Hausdorff directed space are ideals.

Proof of Claim 2: Let T be a path-connected subspace. Suppose there are some $x \in T$ and $y \notin T$ such that $x \to y$. Now $\forall z \in T$, $\exists \gamma$ a path from x to z. $\forall p \in \gamma$, $\exists U_p$ open nbhd of p such that all points in U_p behave the same way with respect to y. Then $\{U_p\}_{p \in \gamma}$ is an open cover of γ, which is compact. So there is a finite subcover $\{U_i\}_{i \in \{1,2,\dots,n\}}$ where $U_1 = U_x, U_n = U_z$. Now as γ is connected, U_1 intersects some other U_i, say U_2. Then with respect to y, U_1 behaves the same as $U_1 \cap U_2$, which behaves the same as U_2. So points in $U_1 \cup U_2$ behaves the same with respect to y. Now again use connectivity, $U_1 \cup U_2$ intersect some other U_i, say U_3. Then by a similar argument points in $U_1 \cup U_2 \cup U_3$ behave the same with respect to y. Keep going, then after finite time we see that the whole γ behave the same with respect to y. So $x \to y$ implies $z \to y$.

Claim 3: In a Hausdorff directed space $(D, \wedge)$, ff $D \setminus \{x\}$ is path-connected for some x, then x is either terminal or initial, i.e. either $x \leftarrow y$ $\forall y$, or $x \to y$ $\forall y$.

Proof of Claim 3: Collapse the ideal $D \setminus \{x\}$ to a point, then the claim becomes trivial.

Claim 4: For path-connected Hausdorff directed space $(D, \wedge)$, $D \setminus \{x\}$ has at most two path components.

Proof of Claim 4: Let T, T', T'' be any distinct three of these components, and collapse them to points t, t', t'' we obtain the space D'. Then $D' \setminus \{t\}$ is path-connected as x is in it to connect other components and t', t'' as well. So t is initial or terminal. Then so does t' and t''. But then there can only be one initial and one terminal, contradiction.

Proof of the Theorem: we only need to show transitivity of "$\to$". Suppose $a \to b$, and $b \to c$ in D. Then b is neither terminal nor initial. So $D \setminus \{b\}$ has two path-connected components. If we collapse them to x, y, then one is initial and the other one is terminal as in Claim 4. Say x is initial and y is terminal. Then a is in the ideal collapsing to x, and c is in the ideal collapsing to y, and clearly $a \to c$ as $x \to y$. □

4 Chicken King Theorems

Theorem 4.1 (Landau)**.** *In any finite directed set (tournament) D, there is an element k (the King) such that $\forall x \in D$, either $k \to x$ or $k \to y$, $y \to x$ for some y.*

Proof. This is a simple proof by induction. Soppose the statement is true for tournament on n vertices. For tournament on $n+1$ vertices, we throw away an arbitrary vertex, call it v_{n+1}, then the rest is a tournament on n vertices, and therefore has a king k. If $k \to v_{n+1}$ as well, or $k \to y$, $y \to v_{n+1}$ for some y, then k is also a king for D. Otherwise, then $\forall x \in D$, if $x = k$, then $v_{n+1} \to x$. If $k \to x$, then as we cannot have $x \to v_{n+1}$, we must have $v_{n+1} \to x$. If $x \to k$, then $\exists y$ such that $k \to y$ and $y \to x$. But then $v_{n+1} \to y$ by last case, so $v_{n+1} \to y$ and $y \to x$. To sum up, v_{n+1} is the king. □

Theorem 4.2 (Nagao, Shakhmatov [1])**.** *Every non-empty compact Hausdorff directed space X has a king.*

Proof. Define $K_x = \{z \in X : z \to y \to x \text{ for some } y \in X\}$. Then we only need to show that $\cap_{x \in X} K_x$ is non-empty. We are going to show that each K_x is closed, and this collection has the finite intersection property, and then by compactness we are done.

We know "$\to$" is a relation on X, so it can also be seen as a subset of $X \times X$. Let's denote this subset G. $\forall (a, b) \notin G$, then $b \to a$. Then as we have shown in the last theorem, $\exists$ disjoint open sets $U \ni a$, $V \ni b$ such that $V \wedge U \subset U$. So $U \times V \subset X \setminus G$. So G is closed.

Then $G \times X$ and $X \times G$ are closed subsets of X^3. Take their intersection, we have a closed subset $T = (a, b, c) : a \to b \to c$. We know $X \times X \times \{x\}$ is also closed for any $x \in X$. So we can define closed subset $Q_x = T \cap (X \times X \times \{x\})$. Let π be the projection onto the first coordinate, then clearly $\pi(Q_x) = K_x$. Now as Q_x is closed in a compact space X^3, it is compact. Then $K_x = \pi(Q_x)$ is compact in a Hausdorff space X, so it is closed.

The finite intersection property of the family $\{K_x\}_{x \in X}$ follows from the original Chicken King Theorem. $\square$

References

[1] Masato Nagao and Dmitri Shakhmatov. On the existence of kings in continuous tournaments, March 2012. arXic:1205.0600[math.GN].